The Generalized Derivative of Exponential Power Functions

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As everyone knows, the derivative method of exponential function $F(x)=u_2(x)^{u_1(x)}$ （$u_2(x)>0$） is more complex, with the help of logarithmic identity or logarithmic derivative method available: $F'(x)=u_1u_2^{u_1-1}u_2'+u_2^{u_1}u_1'\ln u_2$ (1)

This formula is difficult to memorize, seems to have no rules to follow, but through the observation we found an interesting phenomenon, $u_1u_2^{u_1-1}u_2'$ is the derivatives of u_1 which is regarded as constant in $F(x)$, $u_2^{u_1}u_1'\ln u_2$ is the derivatives of u_2 which is regarded as constant in $F(x)$. In general, it can be summed up the following proposition for the generalized power exponential function.

Proposition: Let $F_n(x)=u_n^{u_{n-1}^{\cdot^{\cdot^{\cdot^{u_1}}}}}$, u_1、u_2、$\cdots$、u_n is a x differentiable function, $u_i>0$ $i=2,3,\cdots n$
So $F_n'(x)=f_{n1}+f_{n2}+\cdots+f_{nn}$ （2）, f_{ni} is the derivatives of u_1 when it is only regarded as variable and the others are all regarded as constants in $F(x)$.

Proof：Using mathematical induction, when $n=2$, by（1）, let $n=k$, that is $F_k'(x)=f_{k1}+f_{k2}+\cdots+f_{kk}$ （3）

When $n=k+1$, that is $F_{k+1}(x)=u_{k+1}^{v}$, $v=u_k^{u_{k-1}^{\cdot^{\cdot^{\cdot^{u_1}}}}}$, by（1）

$$F_{k+1}'(x)=u_{k+1}^{v}\,v'\ln u_{k+1}+v\,u_{k+1}^{v-1}u_{k+1}'=u_{k+1}^{u_k^{\cdot^{\cdot^{\cdot^{u_1}}}}}\left(u_k^{u_{k-1}^{\cdot^{\cdot^{\cdot^{u_1}}}}}\right)'\ln u_{k+1}+f_{k+1k+1},$$

If（3） is substituted and induced, $f_{k+1i}=u_{k+1}^{u_k^{\cdot^{\cdot^{\cdot^{u_1}}}}}f_{ki}\ln u_{k+1}$ （$i=1,2,\cdots k$）, so（2）holds.

Proposition holds for the principle of mathematical induction.

Example：For the derivatives of function $f(x)=x^{(x+1)^{\sin x}}$ （$x>0$）

Method: Regarding $(x+1)$ and $\sin x$ as constant, derived $f_1=(x+1)^{\sin x}x^{(x+1)^{\sin x}-1}$

Regarding bottom function x and $\sin x$ as constant, derived $f_2=x^{(x+1)^{\sin x}}(x+1)^{\sin x-1}\sin x\ln x$

Regarding bottom function x and $(x+1)$ as constant, derived $f_3=x^{(x+1)^{\sin x}}(x+1)^{\sin x}\cos x\ln x\ln(x+1)$

So $f'(x)=f_1+f_2+f_3$